A NOTE ON SUMMABILITY METHODS AND SPECTRAL ANALYSIS

BY

CARL S. HERZ[1]

We are concerned with analyzing the spectrum of a bounded measurable function on the real line by means of certain summability methods. If $f \in L^1$, the Fourier transform is $\hat{f}(t) \equiv \int \exp(-itx) f(x)dx$ and $Z(f)$ denotes the set of zeros of $\hat{f}$. Given $\phi \in L^\infty$ we may form the convolution $f \circ \phi(x) \equiv \int f(y)\phi(x-y)dy$. The spectrum of ϕ is defined by $\Lambda(\phi) = \bigcap Z(f)$ where the intersection is taken over all $f \in L^1$ such that $f \circ \phi \equiv 0$.

The underlying heuristic principle of this investigation is that $t \notin \Lambda(\phi)$ if and only if the trigonometric integral $\int \exp(-isx)\phi(x)dx$ is summable, in some suitable sense, to 0 in a neighborhood of t. Since we do not assume that $\int_x^{x+1} |\phi(y)| dy = o(1)$ as $|x| \to \infty$, ordinary convergence is not suitable. Beurling[2] has treated Abel summability and Pollard, [4], the $(R, 2)$ method. However their results may be extended to a large class of summability methods which are quite easy to describe. (Some of our theorems are new even for the Abel and $(R, 2)$ cases.)

DEFINITION. *A function $k \in L^1$ is a "spectral kernel" if $k(0) = 1$ and $k(x) = \int_{|x|}^\infty k'(y)dy$ where $k' \in L^1$.* (We shall consider k' extended to negative arguments as an odd function.)

Set $\Phi_h(t) = \int \exp(-itx)k(hx)\phi(x)dx$. Taking the limit as $h \to 0$ gives a regular summability method.

THEOREM 1. *If k is a spectral kernel, $\Phi_h(t) \to 0$ uniformly in any closed set at positive distance from $\Lambda(\phi)$. More precisely, if t is at a distance δ from $\Lambda(\phi)$, $|\Phi_h(t)| \leqq \delta^{-1} v(\delta h^{-1}) \|\phi\|_\infty$ where v depends only on k and $v(\tau) \to 0$ as $\tau \to \infty$.*

Proof. It suffices to establish the inequality for $t = 0$. We assume the open interval $(-\delta, \delta)$ does not meet $\Lambda(\phi)$. From this it follows, as we shall see in a moment, that if $g \in L^1$ and $\hat{g}(t) = h^{-1}\hat{k}(h^{-1}t)$ in $|t| \geqq \delta$ then $\Phi_h(0) \equiv \int k(hx)\phi(x)dx = \int g(x)\phi(x)dx$. Given $f \in L^1$, let $V_\tau(\hat{f}) = \inf \|g\|_1$, where $\hat{g}(t) = \hat{f}(t)$ in $|t| \geqq \tau$. Clearly $|\Phi_h(0)| \leqq V_\delta\{h^{-1}\hat{k}(h^{-1}t)\} \cdot \|\phi\|_\infty$; however $V_\delta\{h^{-1}\hat{k}(h^{-1}t)\} = h^{-1}V_{\delta h^{-1}}(\hat{k}) = h^{-1}V_{\delta h^{-1}}\{(-it)^{-1}\hat{k}'\}$. By Sz-Nagy's generalization of Bohr's inequality, [5], $V_\tau\{(-it)^{-1}\hat{f}\} \leqq (\pi/2\tau)V_\tau(\hat{f})$ for any $f \in L^1$. Hence $V_\delta\{h^{-1}\hat{k}(h^{-1}t)\} \leqq (\pi/2\delta)V_{\delta h^{-1}}(\hat{k}')$. Also for $f \in L^1$, $V_\tau(\hat{f}) \to 0$ as $\tau \to \infty$.

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(2) Beurling's results are scattered in several papers and in lecture notes. The most relevant of these is [2] where the essential ideas are given.

The statement of the theorem has $v(\tau)=(\pi/2)V_\tau(\hat{k}')$. To see the equality used at the beginning of the proof observe that if $\lambda>1$, the function $\phi(-\lambda x)$ has its spectrum interior to the set $|t|\geqq\delta$. Since the Fourier transform of $f(x)=k(hx)-g(x)$ vanishes on this set, i.e., on a neighborhood of the spectrum of $\phi(-\lambda x)$, it follows from a corollary of Wiener's Tauberian Theorem, see for example [3], that $\int f(x)\phi(\lambda x)dx=0$. Letting λ tend to 1 from above we have the desired equality.

For $t\in\Lambda(\phi)$, $|\Phi_h(t)|\leqq h^{-1}\|k\|_1\|\phi\|_\infty$. Neither this estimate nor the one of Theorem 1 can be improved as can be seen by taking $\phi(x)\equiv1$. However, these estimates are misleading as far as the average behavior is concerned. Parseval's relation yields

$$\left\{(2\pi)^{-1}\int|\Phi_h(t)|^2dt\right\}^{1/2}\leqq h^{-1/2}\|k\|_2\|\phi\|_\infty.$$

The corresponding analogue of Theorem 1 is the simpler result.

THEOREM 2. *Let k be a spectral kernel and Λ_δ the set of points at distance $\geqq\delta$ from $\Lambda(\phi)$. Then $\left\{(2\pi)^{-1}\int_{\Lambda_\delta}|\Phi_h(t)|^2dt\right\}^{1/2}\leqq\delta^{-1/2}w(\delta h^{-1})\|\phi\|_\infty$ where w depends only on k and $w(\tau)\to0$ as $\tau\to\infty$.*

Proof. The reasoning is similar to that of Theorem 1. Suppose $g\in L^1$ and $\hat{g}(t)=h^{-1}\hat{k}(h^{-1}t)$ in $|t|\geqq\delta$. Then if $s\in\Lambda_\delta$, $\Phi_h(s)=\int\exp(-isx)g(x)\phi(x)dx$, so $(2\pi)^{-1}\int_{\Lambda_\delta}|\Phi_h(t)|^2dt\leqq\int|g(x)\phi(x)|^2dx\leqq\|\phi\|_\infty^2\int|g(x)|^2dx$. The infimum of $\int|g(x)|^2dx$ taken over the set specified above is simply $(2\pi)^{-1}\int_{|t|\geqq\delta}|h^{-1}\hat{k}(h^{-1}t)|^2dt$ $=(2\pi h)^{-1}\int_{|t|\geqq\delta h^{-1}}|\hat{k}(t)|^2dt$. Set $w^2(\tau)=\tau\cdot(2\pi)^{-1}\int_{|t|\geqq\tau}|\hat{k}(t)|^2dt$. Then $\left\{(2\pi)^{-1}\int_{\Lambda_\delta}|\Phi_h(t)|^2dt\right\}^{1/2}\leqq\delta^{-1/2}w(\delta h^{-1})\|\phi\|_\infty$, and since $\hat{k}(t)=o(t^{-1})$ as $t\to\infty$, $w(\tau)=o(1)$ as $\tau\to\infty$.

Suppose $f\in L^1$, $\phi\in L^\infty$ and we form the convolution $\psi=f\circ\phi$. There arises the question of the equi-convergence of $\hat{f}(t)\Phi_h(t)$ and $\Psi_h(t)$ $=\int\exp(-itx)k(hx)\psi(x)dx$. A statement equivalent to the first sentence of Theorem 1 is that if T is a closed set and $\hat{f}$ is constant on an ϵ-neighborhood of T then Ψ_h and $\hat{f}\Phi_h$ are uniformly equi-convergent in T. More interesting is the fact that if $(1+|x|)f(x)\in L^1$ and $\lim_{y\to\infty}(2y)^{-1}\int_{-y}^{y}|\phi(x)|dx=0$ then Ψ_h and $f\Phi_h$ are uniformly equi-convergent everywhere. However, no matter how rapidly $|f|$ decreases at ∞, Ψ_h and $f\Phi_h$ need not be equi-convergent everywhere. In particular, let $\phi(x)=i\operatorname{sgn}x$ and let f be arbitrary except that $(1+|x|)f(x)\in L^1$. Then $\lim_{h\to0}\Psi_h(0)-\hat{f}(0)\Phi_h(0)=\lim_{h\to0}\Psi_h(0)=2(d\hat{f}/dt)(0)$, regardless of what spectral kernel is used. Actually this example is completely typical of the type of exceptional behavior which can occur. For $\phi\in L^\infty$ let $\Lambda_0(\phi)$ be the set of points t at which

$$\lim_{j\to\infty}y^{-1}\int_0^y\{\exp(-itx)\phi(x)-\exp(itx)\phi(-x)\}dx\neq0.$$

It is known that every point of $\Lambda_0(\phi)$ is a cluster point of $\Lambda(\phi)$and that $\Lambda_0(\phi)$

has measure zero[3].

THEOREM 3. *Suppose k is a spectral kernel, $\phi \in L^\infty$, $\int(1+|x|)|f(x)|dx<\infty$, and $\psi=f\circ\phi$. Then $\Psi_h-\hat{f}\Phi_h$ is uniformly bounded and $\Psi_h(t)-\hat{f}(t)\Phi_h(t)\to 0$ as $h\to 0$ for all t with the exception of the null set where both $t\in\Lambda_0(\phi)$ and $df/dt\neq 0$.*

Proof. What we actually prove is that $\Psi_h(t)-\hat{f}(t)\Phi_h(t)=-i(d\hat{f}/dt)\cdot F_h(t)+o(1)$ where $F_h(t)=h\int_0^\infty k'(hx)\{\exp(-itx)\phi(x)-\exp(itx)\phi(-x)\}dx$ and o depends only on k', f, and $\|\phi\|_\infty$. By Wiener's Tauberian theorem, $\lim_{h\to 0}F_h(t)=\lim_{y\to\infty}y^{-1}\int_0^y\{\exp(-itx)\phi(x)-\exp(itx)\phi(-x)\}dx$ whenever the limit on the right exists; so the theorem follows directly. For $y\neq 0$, set

$$F_h(t;y)=y^{-1}\int \exp(-itx)\{k(hx)-k[h(x+y)]\}\phi(x)dx.$$

An easy computation shows that

$$\Psi_h(t)-\hat{f}(t)\Phi_h(t)=-\int F_h(t;y)\exp(-ity)yf(y)dy.$$

To conclude the proof we claim $F_h(t;y)-F_h(t)\to 0$ boundedly as $h\to 0$. The calculation is effected by substituting for k in terms of k' in the definition of $F_h(t;y)$, whence

$$\begin{aligned}
|F_h(t;y)-F_h(t)| &= \left|\int\left\{y^{-1}\int_{hx}^{hx+hy}k'(u)du-hk'(hx)\right\}\exp(-itx)\phi(x)dx\right| \\
&\leqq \|\phi\|_\infty\int(hy)^{-1}\int_0^{hy}|k'(x+u)-k'(x)|\,dudx \\
&= \|\phi\|_\infty(hy)^{-1}\int_0^{hy}\int|k'(x+u)-k'(x)|\,dxdu \\
&\leqq 2\|k'\|_1\|\phi\|_\infty
\end{aligned}$$

and $\to 0$ as $h\to 0$ for each y.

Equi-convergence in the mean is a much simpler matter. A trivial generalization of a result of Beurling, [1], is

THEOREM 4. *Suppose k is a spectral kernel, $\phi\in L^\infty$, $\int(1+|x|^{1/2})|f(x)|dx<\infty$, and $\psi=f\circ\phi$. Then $\int|\Psi_h(t)-\hat{f}(t)\Phi_h(t)|^2dt\to 0$ as $h\to 0$.*

Next we turn to some questions in the converse direction from Theorems

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(3) The latter statement follows from an affirmative answer given by W. H. J. Fuchs and others to the author's Research Problem (Bull. Amer. Math. Soc. Research Problem 62-1-2). The solution is unpublished, but the idea is simple. One observes that it is sufficient to take a sequence of y's, say $y=n^2$, in the limit. Then by Plancherel's theorem one has a sequence of functions converging rapidly in the mean to zero. Such a sequence must also converge almost everywhere to zero.

1 and 2. An elementary result using few of the assumptions on k is this one.

THEOREM 5. *If for some* $\epsilon>0$, $\liminf \int_{t-\epsilon}^{t+\epsilon}|\Phi_h(s)|\,ds=0$, *then* $t\notin\Lambda(\phi)$.

Proof. Choose $f\in L^1$ such that $\hat{f}(t)\neq 0$ and $\hat{f}(s)=0$ for $|s-t|\geqq\epsilon$.

$$(2\pi)^{-1}\int \exp(isx)\Phi_h(s)\hat{f}(s)ds = \int k(hy)\phi(y)f(x-y)dy.$$

Since $k(hy)\to 1$ boundedly as $h\to 0$, the right hand side converges to $f\circ\phi$. Hence the left hand side converges, and our assumption ensures that the limit is zero. Thus $f\circ\phi=0$; by definition $\Lambda(\phi)\subset Z(f)$ so $t\notin\Lambda(\phi)$.

The succeeding theorem involves a hypothesis about uniqueness whose verification may be a difficult and delicate matter.

DEFINITION. *The summability kernel k is of type UL^∞ if $\phi\in L^\infty$ and $\Phi_h(t)\to 0$ everywhere imply $\phi=0$ almost everywhere.*

THEOREM 6. *Suppose k is a spectral kernel of type UL^∞ and $\phi\in L^\infty$. If $\Phi_h(s)\to 0$ in a neighborhood of the point t then $t\notin\Lambda(\phi)$.*

Proof. Suppose $\Phi_h(s)\to 0$ in $[t-\epsilon, t+\epsilon]$ for some $\epsilon>0$. By Baire's category theorem, given a sequence $h_n\to 0$, there exist points t_1, t_2, t_3, t_4 with $t-\epsilon<t_1<t_2<t<t_3<t_4<t+\epsilon$ such that $\Phi_{h_n}(s)\to 0$ boundedly for $s\in[t_1, t_2]$ and $s\in[t_3, t_4]$. According to Theorem 5 the intervals (t_1, t_2) and (t_3, t_4) are complementary to $\Lambda(\phi)$. Now choose f such that $\int(1+|x|)|f(x)|dx<\infty$, $\hat{f}(s)=0$ when $s\leqq t_1$ or $s\geqq t_4$, and $\hat{f}(s)=1$ for $t_2\leqq s\leqq t_3$. Put $\psi=f\circ\phi$. The derivative of $\hat{f}$ is 0 on $\Lambda(\phi)$ so by Theorem 3, $\Psi_h(s)\to 0$ everywhere. Since k is of type UL^∞, this implies $\psi\equiv 0$, i.e., $t\notin\Lambda(\phi)$.

Combining Theorems 1 and 6 we have

COROLLARY. *Under the hypotheses of Theorem 6, if $\Phi_h(t)>0$ in an open interval, the convergence is uniform in each closed subinterval.*

For a discussion of ordinary convergence, $k(x)=1$ for $|x|\leqq 1$, $=0$ for $|x|>1$, see [6]. In this case modifications of all the preceding theorems are valid under the hypothesis $\int_x^{x+1}|\phi(y)|dy=o(1)$ as $|x|\to\infty$. More generally, under some appropriate assumption of the form "ϕ is asymptotically small," e.g., $\phi(x)=o(1)$ or $\phi\in L^q$, $q<\infty$, one can replace the requirement that k be a spectral kernel by the conditions: $k\in L^1\cap L^\infty$, $k(0)=1$, k is continuous at 0, and $\int|k(x+y)-k(x)|dx=O(y)$ as $y\to 0$. Then Theorem 3 holds in the form: $\Psi_h(t)-\hat{f}(t)\Phi_h(t)\to 0$ uniformly, while the proof of Theorem 4 is unmolested. Less precise versions of Theorems 1 and 2 may be derived from Theorems 3 and 4 respectively.

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CORNELL UNIVERSITY,
ITHACA, N. Y.